A new H2-norm Lyapunov function for the stability of a singularly perturbed system of two conservation laws

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Abstract— **In this paper a class of singularly perturbed system of conservation laws is considered. The partial differential equations are equipped with boundary conditions which may be studied to derive the exponential stability. Lyapunov stability technique is used to derive sufficient conditions for the exponential stability of this system. A Lyapunov function in H^2-norm for a singularly perturbed system of conservation laws is constructed. It is based on the Lyapunov functions of two subsystems in L^2-norm.**

I. INTRODUCTION

The singular perturbation techniques occurred at the beginning of the 20th century. A great deal of the early motivation in this area arose from the studies of physical problems exhibiting both fast and slow dynamics, for instance DC-motor model, voltage regulator in [1] and semiconducting diode in [2]. The development of this method led to the efficient use in various fields in mathematical physics and engineering, for example, fluid mechanics, fluid dynamics, elasticity, quantum mechanics, chemical-reactor, aerodynamics etc. (see [3] for a survey).

The stability properties of singularly perturbed systems have been investigated by several authors. In the papers [4], [5], [6], [7] and [8], Lyapunov method, which is commonly used for stability analysis of dynamical systems, has been employed. The main idea is to consider two subsystems: the reduced system representing the slow dynamic and the boundary-layer system standing for the fast dynamic. Assuming that each of the two subsystems is stable and has a Lyapunov function, the stability of the singularly perturbed system can be established by a Lyapunov function which is obtained by the composition of the Lyapunov function of the reduced and boundary-layer systems, for a sufficiently small perturbation parameter. As in [8], a composite Lyapunov function has been investigated for the asymptotic stability of a singularly perturbed finite-dimensional nonlinear system.

In our previous work [9], a class of singularly perturbed system of two conservation laws with a small perturbation parameter introduced to both the dynamics and the boundary conditions has been studied. As soon as the two subsystems, the reduced and boundary-layer systems, are stable, the stability of this system can be obtained by a Lyapunov function in L^2-norm. The present paper focuses on the stability problem of the singularly perturbed system of two conservation laws with a small perturbation parameter ε introduced only to the dynamics. The first problem under our consideration is the stability of the two subsystems. Some necessary and sufficient conditions are stated for exponential stability of these subsystems. Each of the two subsystems has a strict Lyapunov function in L^2-norm. Furthermore, we consider the exponential stability of the whole singularly perturbed system. For a sufficiently small perturbation parameter, with some additional conditions on the boundary conditions, the stability of the whole singularly perturbed system can be established via strict Lyapunov function in H^2-norm. The new element here is the use of Lyapunov function in H^2-norm. This kind of Lyapunov function has been studied in several papers. In [10] a strict H^2-norm Lyapunov function has been constructed to analyze the stability of solutions to a system of two hyperbolic conservation laws around equilibrium. The stability of one-dimensional $n \times n$ nonlinear hyperbolic systems has also been considered in [11]. And in the work of [12], it is concerned with H^2- stabilization of the Isothermal Euler equations.

The paper is organized as follows. The class of singularly perturbed system of conservation laws under consideration in this paper is given in Section 2. Section 3 states the exponential stability of the reduced and boundary-layer systems via Lyapunov functions in L^2-norm. Section 4 analyzes the stability for the overall singularly perturbed system. A strict Lyapunov function in H^2-norm is constructed. In Section 5, an illustrative example is provided to show the main result. Finally, concluding remarks end the paper. Some proofs have been omitted due to space limitation.

Notation. For a partitioned symmetric matrix P, the symbol $\bigstar$ stands for symmetric block, $P \geqslant 0$ means that P is positive semidefinite. Given a matrix A, the transpose matrix of A is denoted as A^T. The associate norm in $H^2(0,1)$ space is denoted by $||\boldsymbol{.}||_{H^2}$, defined for all functions $f \in H^2(0,1)$, by

$$||f||_{H^2} = \left(\int_0^1 f^2 + f_x^2 + f_{xx}^2 dx\right)^{1/2}. \tag{1}$$

Following [11], we introduce the notation, for all matrices $M \in \mathbb{R}^{n\times n}$,

$$\rho_1(M) = \inf\{\left|\left|\Delta M \Delta^{-1}\right|\right|, \Delta \in D_{n,+}\}, \tag{2}$$

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where $||\,.\,||$ denotes the usual matrix 2-norm and $\Delta \in D_{n,+}$ denotes the set of diagonal positive matrix in $\mathbb{R}^{n\times n}$.
According to [13], for all matrices $M \in \mathbb{R}^{n\times n}$, $\rho(M)$ is defined as

$$\rho(M) = \inf\{|||\Delta M \Delta^{-1}|||, \Delta \in D_{n,+}\}; \tag{3}$$

where

$$|||M||| = \max\left\{\sum_{j=1}^{n} |M_{ij}|; i \in 1,\dots,n\right\}. \tag{4}$$

II. PROBLEM FORMULATION

We consider the following singularly perturbed system of conservation laws for a small positive perturbation parameter ε:

$$\begin{aligned} y_t(x,t) + y_x(x,t) &= 0, \\ \varepsilon z_t(x,t) + z_x(x,t) &= 0, \end{aligned} \tag{5}$$

where $x \in [0,1]$, $t \in [0,+\infty)$, $y : [0,1]\times[0,+\infty) \to \mathbb{R}$, $z : [0,1]\times[0,+\infty) \to \mathbb{R}$, with the boundary conditions

$$\begin{pmatrix} y(0,t) \\ z(0,t) \end{pmatrix} = G \begin{pmatrix} y(1,t) \\ z(1,t) \end{pmatrix}, \tag{6}$$

where $G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$ is a 2×2 constant matrix.

Given two continuous functions $y^0 : [0,1] \to \mathbb{R}$ and $z^0 : [0,1] \to \mathbb{R}$, the initial conditions are:

$$\begin{pmatrix} y(x,0) \\ z(x,0) \end{pmatrix} = \begin{pmatrix} y^0 \\ z^0 \end{pmatrix}. \tag{7}$$

Let us compute the reduced and boundary-layer systems following the approach of [1]. By setting $\varepsilon = 0$ in system (5), the reduced system is defined as:

$$y_t(x,t) + y_x(x,t) = 0, \tag{8a}$$
$$z_x(x,t) = 0. \tag{8b}$$

Substitute (8b) into the boundary condition (6) yields:

$$\begin{aligned} y(0,t) &= \left(G_{11} + \tfrac{G_{12}G_{21}}{1-G_{22}}\right) y(1,t), \\ z(.,t) &= \tfrac{G_{21}}{1-G_{22}} y(1,t). \end{aligned} \tag{9}$$

The reduced system is rewritten as

$$\bar{y}_t(x,t) + \bar{y}_x(x,t) = 0, \tag{10}$$

with the boundary condition

$$\bar{y}(0,t) = \left(G_{11} + \frac{G_{12}G_{21}}{1-G_{22}}\right) \bar{y}(1,t). \tag{11}$$

To define the boundary-layer system, the variable $y(x,t)$ is seen as a constant with respect to time, which yields:

$$\bar{z}_\tau(x,\tau) + \bar{z}_x(x,\tau) = 0, \tag{12}$$

with the boundary condition:

$$\bar{z}(0,\tau) = G_{21}\bar{y}(1) + G_{22}\bar{z}(1,\tau), \tag{13}$$

where $\tau = \frac{t}{\varepsilon}$ is a stretching time scale. $\bar{y}(1)$ in (13) is handled as a fixed parameter for the boundary-layer system.

Remark. According to Proposition 2.1 in [11], for every $\begin{pmatrix} y^0 \\ z^0 \end{pmatrix} \in H^2(0,1)$ satisfying the following compatibility conditions:

$$\begin{pmatrix} y^0(0) \\ z^0(0) \end{pmatrix} = G \begin{pmatrix} y^0(1) \\ z^0(1) \end{pmatrix}, \tag{14}$$
$$\begin{pmatrix} y^0_x(0) \\ \frac{1}{\varepsilon} z^0_x(0) \end{pmatrix} = G \begin{pmatrix} y^0_x(1) \\ \frac{1}{\varepsilon} z^0_x(1) \end{pmatrix}, \tag{15}$$

the system (5) and (6) has a unique maximal classical solution $\begin{pmatrix} y \\ z \end{pmatrix} \in C^0([0,+\infty), H^2(0,1))$.

III. STABILITY ANALYSIS OF REDUCED AND BOUNDARY-LAYER SYSTEMS

The first problem under consideration is the stability analysis of the reduced system (10) and (11) and the boundary-layer system (12) and (13). Our objective in this section is to establish the stability properties of the reduced and boundary-layer systems via strict Lyapunov function in L^2-norm.

Definition 1: *The reduced system (10) and (11) is exponentially stable in L^2-norm if there exist $\alpha > 0$ and $C > 0$ such that, for every $\bar{y}^0 \in L^2(0,1)$, the solution to the reduced system (10) and (11) satisfies $||\bar{y}(.,t)||_{L^2} \leqslant Ce^{-\alpha t}||\bar{y}^0||_{L^2}$.*

Proposition 1: *The reduced system (10) and (11) is exponentially stable in L^2-norm to 0 if and only if the boundary condition satisfies*

$$\left| G_{11} + \frac{G_{12}G_{21}}{1-G_{22}} \right| < 1. \tag{16}$$

Under this condition, a strict Lyapunov function is defined as:

$$V(\bar{y}) = \int_0^1 e^{-\mu x}\bar{y}^2 dx, \tag{17}$$

for all $\bar{y} \in L^2(0,1)$, where $\mu > 0$ satisfies

$$e^{-\mu} \geqslant \left(G_{11} + \frac{G_{12}G_{21}}{1-G_{22}}\right)^2. \tag{18}$$

Proposition 2: *Given any $\bar{y}(1)$ the boundary-layer system (12) and (13) is exponentially stable in L^2-norm to the equilibrium point $\frac{G_{21}}{1-G_{22}}\bar{y}(1)$ if and only if the boundary condition satisfies*

$$|G_{22}| < 1. \tag{19}$$

Under this condition, a strict Lyapunov function is defined as:

$$W(\bar{z}) = \int_0^1 e^{-\nu x}\left(\bar{z} - \frac{G_{21}}{1-G_{22}}\bar{y}(1)\right)^2 dx, \tag{20}$$

for all $\bar{z} \in L^2(0,1)$, where $\nu > 0$ satisfies

$$e^{-\nu} \geqslant G_{22}^2. \tag{21}$$

IV. STABILITY ANALYSIS OF SINGULARLY PERTURBED SYSTEM OF CONSERVATION LAWS

The aim of this section is to state the exponential stability of the singularly perturbed system of conservation laws from that of the reduced and boundary-layer systems, for small $\varepsilon > 0$.

In the previous section, the exponential stability has been established via Lyapunov functions in L^2-norm for the reduced and boundary-layer systems. As we will see later, in order to do the stability analysis of the overall singularly perturbed system of conservation laws, it is needed to consider a strict Lyapunov function in H^2-norm.

Definition 2: *The singularly perturbed system of conservation laws (5) and (6) is exponentially stable to the origin in H^2-norm if there exist $\gamma > 0$ and $C_1 > 0$, for every $\begin{pmatrix} y^0 \\ z^0 \end{pmatrix} \in H^2(0,1)$, the solution to the system (5) and (6) satisfies $(||y(.,t)||_{H^2} + ||z(.,t)||_{H^2}) \leqslant C_1 e^{-\gamma t} \left(||y^0||_{H^2} + ||z^0||_{H^2}\right)$.*

Our result shows that as soon as the reduced system (10) and (11) and the boundary-layer system (12) and (13) are exponentially stable in L^2-norm, with some additional conditions on the boundary conditions, for sufficiently small ε, the exponential stability of singularly perturbed system of conservation laws (5) and (6) holds in H^2-norm.

Let us start by stating our assumptions.

Assumption 1: *The reduced system (10) with boundary condition (11) is exponentially stable in L^2-norm.*

Assumption 2: *The boundary-layer system (12) with boundary condition (13) is exponentially stable in L^2-norm.*

The last assumption (Assumption 3) deals with technical properties on matrix G in the boundary condition (6).

Assumption 3: *For a given d such that $0 < d < 1$, $\mu > 0$ (resp. $\nu > 0$) such that $e^{-\mu} > G_{11}^2$ and $e^{-\mu} > \left(G_{11} + \frac{G_{12}G_{21}}{1-G_{22}}\right)^2$ (resp. $e^{-\nu} > G_{22}^2$), the following three inequalities hold:*

a)

$$(1-d)R_1 - dG_{21}^2 \quad \geqslant \quad 0, \tag{22}$$

b)

$$dR_2 - (1-d)G_{12}^2 \quad \geqslant \quad 0, \tag{23}$$

c)

$$\begin{aligned} &\left((1-d)R_1 - dG_{21}^2\right)\left(dR_2 - (1-d)G_{12}^2\right) \\ &- \left((1-d)R_3 + dR_4\right)^2 \geqslant 0. \end{aligned} \tag{24}$$

where:
$R_1 = \left(e^{-\mu} - G_{11}^2\right)$, $R_2 = \left(e^{-\nu} - G_{22}^2\right)$, $R_3 = G_{11}G_{12}$, $R_4 = G_{21}G_{22}$.

Remark. 1) Due to Proposition 1, Assumption 1 is equivalent to $\left|G_{11} + \frac{G_{12}G_{21}}{1-G_{22}}\right| < 1$, which guarantees that $y = 0$ is an exponentially stable equilibrium point of the reduced system (10) and (11).
2) Similarly, due to Proposition 2, Assumption 2 is equivalent to $|G_{22}| < 1$, which plays the same role for the boundary-layer system (12) and (13) which has an equilibrium point $z = \frac{G_{21}}{1-G_{22}}y(1)$. It is important to notice that, for the boundary-layer system, $y(1)$ is treated as a constant with respect to time. On the other hand it is reasonable to consider $y(1)$ as a variable of time for the overall system (5) and (6).
3) Assumption 3 stands for the additional conditions on the boundary conditions, which ensures the solutions converge in H^2-norm.

We can state the following theorem:

Theorem 1: *Under Assumptions 1, 2 and 3, let the positive value $\varepsilon^* = min(\varepsilon_1, \varepsilon_2)$, $\varepsilon_1, \varepsilon_2 \in (0, +\infty]$ be given by:*
Case 1: If $G_{12} \neq 0$ or $G_{21} \neq 0$

$$\varepsilon_1 \quad = \quad \frac{dK_1R_2}{(1-d)G_{12}^2K_1 + (1-d)K_2^2}; \tag{25}$$

$$\varepsilon_2 \quad = \quad \frac{(1-d)\mu e^{-\mu}\nu e^{-\nu}(1-G_{22})^2}{2dG_{21}^2}. \tag{26}$$

where: $K_1 = e^{-\mu} - \left(G_{11} + \frac{G_{12}G_{21}}{1-G_{22}}\right)^2$ and $K_2 = \left(G_{11}G_{12} + \frac{G_{12}^2G_{21}}{1-G_{22}}\right)$.

Case 2: If $G_{12} = 0$ and $G_{21} = 0$

$$\varepsilon_1 = \varepsilon_2 = +\infty, \tag{27}$$

Then, for all $0 < \varepsilon < \varepsilon^*$ and $0 < \varepsilon < +\infty$, the singularly perturbed system of conservation laws (5) and (6) is exponentially stable in H^2-norm to the origin and it has a strict Lyapunov function:

$$\begin{aligned} &L(y,z) \\ &= (1-d)\int_0^1 e^{-\mu x}(y^2 + y_x^2 + y_{xx}^2)dx + \\ &d\int_0^1 e^{-\nu x}\left(\left(z - \frac{G_{21}}{1-G_{22}}y(1)\right)^2 + \eta_1(\varepsilon)z_x^2 + \eta_2(\varepsilon)z_{xx}^2\right) \end{aligned} \tag{28}$$

where η_1, η_2 are positive values which depend on ε.

Remark. ε^* can be computed by ε_1 and ε_2, moreover, to ensure the stability of system (5) and (6) in H^2-norm for all $\varepsilon < \varepsilon^*$, ε^* should be taken as the minimum value between ε_1 and ε_2.

Sketch of the proof: First, let us decompose $L(y,z)$ in the following way:

$$L(y,z) \quad = \quad L_1 + L_2 + L_3, \tag{29}$$

with L_1, L_2 and L_3 selected respectively to the zero-th, first and second space-derivative of the solutions, that is:

$$\begin{aligned} L_1 = (1-d)&\int_0^1 e^{-\mu x}y^2dx \\ &+d\int_0^1 e^{-\nu x}\left(z - \frac{G_{21}}{1-G_{22}}y(1)\right)^2dx, \end{aligned} \tag{30}$$

$$L_2 \quad = \quad (1-d)\int_0^1 e^{-\mu x}y_x^2dx + d\eta_1(\varepsilon)\int_0^1 e^{-\nu x}z_x^2dx, \tag{31}$$

$$L_3 = (1-d)\int_0^1 e^{-\mu x} y_{xx}^2 dx + d\eta_2(\varepsilon)\int_0^1 e^{-\nu x} z_{xx}^2 dx. \tag{32}$$

Next, we use 4 steps to demonstrate the proof of Theorem 1.

Step 1: Compute the time derivative of the first term L_1 along the solutions to (5) and (6)

$$\dot{L_1} = L_{11} + L_{12} \tag{33}$$

with

$$\begin{aligned} L_{11} &= -(1-d)\left[e^{-\mu x} y^2\right]_{x=0}^{x=1} \\ &\quad -\frac{d}{\varepsilon}\left[e^{-\nu x}\left(z - \frac{G_{21}}{1-G_{22}} y(1)\right)^2\right]_{x=0}^{x=1}, \end{aligned} \tag{34}$$

and

$$\begin{aligned} L_{12} &= -(1-d)\mu\int_0^1 e^{-\mu x} y^2 dx + \left(\frac{2dG_{21}}{1-G_{22}}\right) \\ &\quad \int_0^1 e^{-\nu x}\left(z - \frac{G_{21}}{1-G_{22}} y(1)\right) y_x(1) dx \\ &\quad -\frac{d}{\varepsilon}\nu\int_0^1 e^{-\nu x}\left(z - \frac{G_{21}}{1-G_{22}} y(1)\right)^2 dx. \end{aligned} \tag{35}$$

Under the boundary conditions (6) and replacing $z(1)$ by the right-hand side of the following equation

$$z(1) = \left(z(1) - \frac{G_{21}}{1-G_{22}} y(1)\right) + \frac{G_{21}}{1-G_{22}} y(1),$$

the term L_{11} follows

$$L_{11} = -\begin{pmatrix} y(1) \\ z(1) - \frac{G_{21}}{1-G_{22}} y(1) \end{pmatrix}^T M_{11} \begin{pmatrix} y(1) \\ z(1) - \frac{G_{21}}{1-G_{22}} y(1) \end{pmatrix}$$

with

$$M_{11} = \begin{pmatrix} (1-d)K_1 & -(1-d)K_2 \\ \bigstar & \frac{d}{\varepsilon}R_2 - (1-d)G_{12}^2 \end{pmatrix},$$

where K_1, K_2 are defined in Theorem 1 and R_2 is defined in Assumption 3.
To prove L_{11} is non positive, which is equivalent to prove the matrix $M_{11} \geqslant 0$, it is sufficient to require that

$$0 < \varepsilon \leqslant \frac{dK_1R_2}{(1-d)G_{12}^2K_1 + (1-d)K_2^2}, \tag{36}$$

if $G_{12} \neq 0$. And pick any $\varepsilon > 0$ if $G_{12} = 0$.
Due to Assumption 1 (resp. Assumption 2), K_1 (resp. R_2) is always positive. The right-hand side of (36) gives ε_1.

Step 2: Differentiating (5) with respect to x, we have

$$\begin{aligned} y_{xt}(x,t) + y_{xx}(x,t) &= 0, \\ \varepsilon z_{xt}(x,t) + z_{xx}(x,t) &= 0, \end{aligned} \tag{37}$$

with the boundary conditions

$$\begin{aligned} y_x(0,t) &= G_{11}y_x(1,t) + \tfrac{1}{\varepsilon}G_{12}z_x(1,t), \\ z_x(0,t) &= \varepsilon G_{21}y_x(1,t) + G_{22}z_x(1,t). \end{aligned} \tag{38}$$

Compute the time derivative of the second term L_2 along the solutions to (37) and (38)

$$\dot{L_2} = L_{21} + L_{22} \tag{39}$$

with

$$L_{21} = -(1-d)[e^{-\mu x}y_x^2]_{x=0}^{x=1} - \frac{d\eta_1(\varepsilon)}{\varepsilon}[e^{-\nu x}z_x^2]_{x=0}^{x=1}, \tag{40}$$

and

$$L_{22} = -(1-d)\mu\int_0^1 e^{-\mu x}y_x^2 dx - \frac{d\nu\eta_1(\varepsilon)}{\varepsilon}\int_0^1 e^{-\nu x}z_x^2 dx. \tag{41}$$

Take $\eta_1(\varepsilon) = \frac{1}{\varepsilon}$, under the boundary conditions (38), the term L_{21} follows

$$L_{21} = -\begin{pmatrix} y_x(1) \\ z_x(1) \end{pmatrix}^T M_{21} \begin{pmatrix} y_x(1) \\ z_x(1) \end{pmatrix},$$

with:

$$M_{21} = \begin{pmatrix} (1-d)R_1 - dG_{21}^2 & -(\frac{(1-d)R_3}{\varepsilon} + \frac{dR_4}{\varepsilon}) \\ \bigstar & \frac{d}{\varepsilon^2}R_2 - \frac{(1-d)G_{12}^2}{\varepsilon^2} \end{pmatrix},$$

where R_1, R_2, R_3 and R_4 are defined in Assumption 3.
To prove L_{21} is non positive, which is equivalent to prove the matrix $M_{21} \geqslant 0$, it is sufficient to require the following conditions are satisfied:

$$(1-d)R_1 - dG_{21}^2 \geqslant 0; \tag{42}$$

it is equivalent to the first inequality of Assumption 3.

$$\frac{d}{\varepsilon^2}R_2 - \frac{(1-d)G_{12}^2}{\varepsilon^2} \geqslant 0; \tag{43}$$

it is equivalent to the second inequality of Assumption 3.

$$\begin{aligned} &\frac{\left((1-d)R_1 - dG_{21}^2\right)\left(dR_2 - (1-d)G_{12}^2\right)}{\varepsilon^2} \\ &-\frac{\left((1-d)R_3 + dR_4\right)^2}{\varepsilon^2} \geqslant 0; \end{aligned} \tag{44}$$

it is equivalent to the third inequality of Assumption 3.
Step 3: Differentiating (37) with respect to x, we have:

$$\begin{aligned} y_{xxt}(x,t) + y_{xxx}(x,t) &= 0, \\ \varepsilon z_{xxt}(x,t) + z_{xxx}(x,t) &= 0, \end{aligned} \tag{45}$$

with boundary conditions:

$$\begin{aligned} y_{xx}(0,t) &= G_{11}y_{xx}(1,t) + \tfrac{1}{\varepsilon^2}G_{12}z_{xx}(1,t), \\ z_{xx}(0,t) &= \varepsilon^2 G_{21}y_{xx}(1,t) + G_{22}z_{xx}(1,t). \end{aligned} \tag{46}$$

Compute the time derivative of the third term L_3 along the solutions to (45) and (46)

$$\dot{L_3} = L_{31} + L_{32} \tag{47}$$

with

$$L_{31} = -(1-d)[e^{-\mu x}y_{xx}^2]_{x=0}^{x=1} - \frac{d\eta_2(\varepsilon)}{\varepsilon}[e^{-\nu x}z_{xx}^2]_{x=0}^{x=1}, \tag{48}$$

and

$$L_{32} = -(1-d)\mu \int_0^1 e^{-\mu x} y_{xx}^2 dx - \frac{d\nu\eta_2(\varepsilon)}{\varepsilon} \int_0^1 e^{-\nu x} z_{xx}^2 dx. \tag{49}$$

Take $\eta_2(\varepsilon) = \frac{1}{\varepsilon^3}$, under the boundary conditions (46), the term L_{31} follows

$$L_{31} = -\begin{pmatrix} y_{xx}(1) \\ \frac{z_{xx}(1)}{\varepsilon} \end{pmatrix}^T M_{21} \begin{pmatrix} y_{xx}(1) \\ \frac{z_{xx}(1)}{\varepsilon} \end{pmatrix}.$$

As soon as the matrix $M_{21} \geqslant 0$, L_{31} is non positive.

Step 4: Combining all the integral terms (35), (41) and (49), with $\eta_1(\varepsilon) = \frac{1}{\varepsilon}$, $\eta_2(\varepsilon) = \frac{1}{\varepsilon^3}$, we get

$$\begin{aligned} & L_{12} + L_{22} + L_{32} \\ &= -(1-d)\mu \int_0^1 e^{-\mu x}(y^2 + y_x^2 + y_{xx}^2) dx \\ &\quad -\frac{d\nu}{\varepsilon} \int_0^1 e^{-\nu x} \left(\left(z - \frac{G_{21}}{1-G_{22}} y(1) \right)^2 + \frac{z_x^2}{\varepsilon} + \frac{z_{xx}^2}{\varepsilon^3} \right) \\ &\quad + \left(\frac{2dG_{21}}{1-G_{22}} \right) \int_0^1 e^{-\nu x} \left(z - \frac{G_{21}}{1-G_{22}} y(1) \right) y_x(1) dx. \end{aligned} \tag{50}$$

The following inequalities hold

$$\begin{aligned} & -(1-d)\mu \int_0^1 e^{-\mu x}(y^2 + y_x^2 + y_{xx}^2) dx \\ \leqslant \; & -(1-d)\mu e^{-\mu} \|y\|_{H^2}^2. \end{aligned} \tag{51}$$

Moreover

$$\begin{aligned} & -\frac{d\nu}{\varepsilon} \int_0^1 e^{-\nu x} \left(\left(z - \frac{G_{21}}{1-G_{22}} y(1) \right)^2 + \frac{z_x^2}{\varepsilon} + \frac{z_{xx}^2}{\varepsilon^3} \right) dx \\ & \leqslant -\frac{d}{\varepsilon} \nu e^{-\nu} \| z - \frac{G_{21}}{1-G_{22}} y(1) \|_{H^2}^2. \end{aligned} \tag{52}$$

Furthermore

$$\begin{aligned} & \frac{2dG_{21}}{1-G_{22}} \int_0^1 e^{-\nu x} \left(z - \frac{G_{21}}{1-G_{22}} y(1) \right) y_x(1) dx \\ \leqslant \; & \left| \frac{2\sqrt{2} dG_{21}}{1-G_{22}} \right| \left\| z - \frac{G_{21}}{1-G_{22}} y(1) \right\|_{H^2} \|y\|_{H^2}. \end{aligned} \tag{53}$$

According to (51), (52) and (53), $L_{12} + L_{22} + L_{32}$ follows

$$\begin{aligned} & L_{12} + L_{22} + L_{32} \\ & \leqslant -\begin{pmatrix} \|y\|_{H^2} \\ \|(z - \frac{G_{21}}{1-G_{22}} y(1)\|_{H^2} \end{pmatrix}^T M_4 \begin{pmatrix} \|y\|_{H^2} \\ \|(z - \frac{G_{21}}{1-G_{22}} y(1)\|_{H^2} \end{pmatrix}, \end{aligned}$$

with

$$M_4 = \begin{pmatrix} (1-d)\mu e^{-\mu} & -|\frac{\sqrt{2} dG_{21}}{1-G_{22}}| \\ \bigstar & \frac{d\nu}{\varepsilon} e^{-\nu} \end{pmatrix}.$$

To prove $M_4 > 0$, it is sufficient to require that

$$0 < \varepsilon < \frac{(1-d)\mu e^{-\mu} \nu e^{-\nu} (1-G_{22})^2}{2dG_{21}^2}. \tag{54}$$

Let ε_2 be given by the right-hand side of (54).

Under Assumptions 1, 2 and 3, pick $\varepsilon^* = min(\varepsilon_1, \varepsilon_2)$, where ε_1 and ε_2 are given by (25) and (26) in Case 1 and (27) in Case 2. λ is the minimal eigenvalue of M_4, it is obtained:

$$\dot{L} \leqslant -\lambda \left(\left\| z - \frac{G_{21}}{1-G_{22}} y(1) \right\|_{H^2}^2 + \|y\|_{H^2}^2 \right). \tag{55}$$

Therefore the Lyapunov function (28) is a strict Lyapunov function for singularly perturbed system of conservation laws (5) and (6). This concludes the proof of Theorem 1. ∎

V. NUMERICAL EXAMPLE

In this section, we consider the following boundary condition for the singularly perturbed system (5):

$$G = \begin{pmatrix} 0.7 & 2 \\ 0.1 & -0.5 \end{pmatrix}, \tag{56}$$

where Assumptions 1 and 2 hold. Let take $d = 0.95$, $\mu = \nu = 0.1$, it is computed: $R_1 = 0.4148$, $R_2 = 0.6548$, $R_3 = 1.4$, $R_4 = -0.05$, the three inequalities of Assumption 3 hold. Following the statement of Theorem 1 it is computed: $K_1 = 0.2104$, $K_2 = 1.6666$. According to (25) and (26), ε_1 and ε_2 are computed:

$$\begin{aligned} \varepsilon_1 &= 0.7233, \\ \varepsilon_2 &= 0.0485. \end{aligned}$$

The admissible perturbation parameter ε is chosen as 0.045.

Applying Theorem 1, the system (5) and (56) is exponentially stable. Considering a diagonal positive definite matrix $\Delta = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$, the inequality $||\Delta G \Delta^{-1}|| < 1$ holds and thus $\rho_1(G) < 1$. Therefore with [11], we recover the stability of system (5) and (56). In other words, the stability condition of [11] applies for system (5) and (56). However $\rho(|G|) > 1$, and thus the stability condition of [13] does not apply.

Let us check the stability in numerical simulation of (5) and (56). Discretize the equation (5) using a two-step variant of the Lax-Wendroff method which is presented in [14] and the solver on Matlab in [15]. More precisely, we divide the space domain $[0, 1]$ into 100 intervals of identical length, and 15 as final time. We choose a time-step that satisfies the CFL condition for the stability and select the following initial functions:

$$\begin{aligned} y(x, 0) &= \sin(4\pi x) \\ z(x, 0) &= \sin(5\pi x) \end{aligned}$$

for all $x \in [0, 1]$.

The time evolutions of the solutions $\begin{pmatrix} y \\ z \end{pmatrix}$ and of square of H^2-norm $\begin{pmatrix} ||y||_{H^2}^2 \\ \left|\left| z - \frac{G_{21}}{1-G_{22}} y(1) \right|\right|_{H^2}^2 \end{pmatrix}$ are in Figures 1 and

2 respectively. It is observed that the two components y and z converge to 0 as time increases from Figure 1, and the z-component of the solution converges faster than the y-component of the solution from Figure 2.

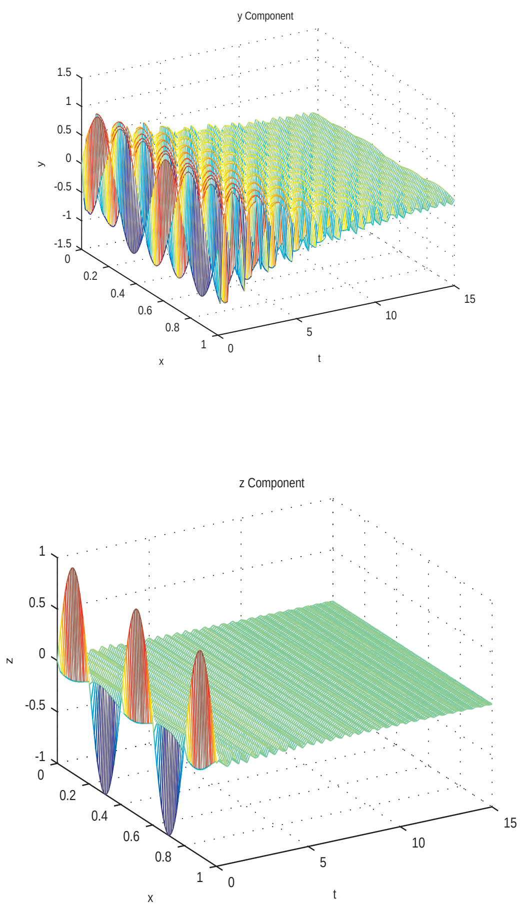

Fig. 1. Time evolutions of the first component y (top) and of the second component z (bottom) of the solution of the system (5) and (56)

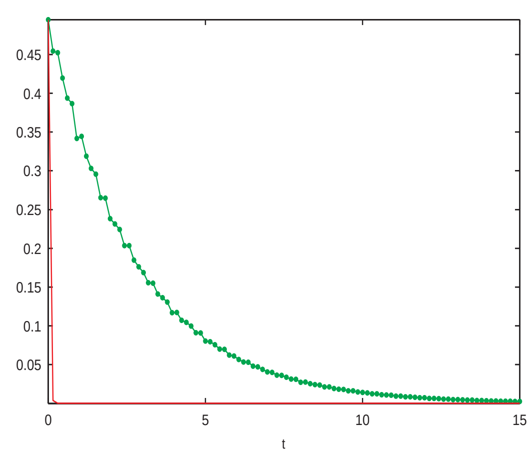

Fig. 2. Time evolutions of square of H^2-norm $||y||_{H^2}^2$ (green dotted line) and square of H^2-norm $\left|\left|z - \frac{G_{21}}{1-G_{22}} y(1)\right|\right|_{H^2}^2$(red plain line)

VI. CONCLUSIONS

In this paper, the stability for a class of singularly perturbed system of conservation laws has been studied. Necessary and sufficient condition has been derived for the stability of the reduced and boundary-layer systems. For a perturbation parameter sufficiently small, under suitable assumptions on the boundary conditions, the exponential stability of the singularly perturbed system of conservation laws has been established in H^2-norm. This result was proved by means of a Lyapunov function approach.

This work was restricted to analyze singularly perturbed system of two conservation laws. It is natural to extend to the system of balanced laws and to systems of higher dimension. Another interesting point is to consider some physical application, like gas flow through pipelines in [16], [17] and open channels in [18].

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